The effect of a very strong magnetic cross-field on steady motion through a slightly conducting fluid

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The flow engendered by the steady motion of a cylindrical insulator through an inviscid, incompressible fluid of small conductivity σ is not close to potential flow when the applied magnetic cross-field H_0 is sufficiently strong. Here we determine the limiting form of this flow as $\sigma \to 0$ with $\sigma H_0^2 \to \infty$, the latter representing the ponderomotive force.

The limit equations do not have a unique solution, but it is possible to make a selection by taking into account the inertia of the fluid during the limiting process, i.e. without recourse to considerations of how the motion was set up from rest. The forces on the cylinder are found to be asymptotically proportional to $\sqrt{\sigma} H_0$.

The case of an elliptic cylinder and that of a flat plate are worked out in detail.

1. Introduction

Recently J. D. Murray and the author (1960) investigated the flow of a slightly conducting, incompressible, inviscid fluid past a fixed obstacle. The applied magnetic field was assumed to be weak (in fact, it need only be moderate) and to originate in the body itself. The same methods work for a uniform applied field.

In the present paper the fluid is again taken to be a bad conductor, but now the ambient field is assumed to be very strong, being applied at infinity at right angles to the free-stream direction. For simplicity we consider plane flow past a cylindrical insulator of the same permeability as the fluid, perpendicular to both the free stream and the applied magnetic field.

Because of the small conductivity σ, the relative change in the magnetic field $\mathbf{H}_0$, due to motion-induced currents, is small; in the previous problem this disturbance was nevertheless of some importance, but here it may safely be neglected. On the other hand, the fluid motion is no longer close to the potential flow which occurs for $\sigma = 0$. If it were, there would be induced currents of order σH_0—which may still be small—on which would act ponderomotive forces of order σH_0^2, and it is this last quantity which we take to be laıge in speaking of a very strong magnetic field. This contradiction can only be avoided by assuming a completely different limit flow.

In fact, as $\sigma \to 0$ with $\sigma H_0^2 \to \infty$, the flow attains a rigidity in which only the component of velocity, v, along the lines of force can be disturbed and all quantities are unvarying in this direction (y).† However, there are many such solutions

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† Fig. 1 gives one such flow, which is not, however, the correct one (§ 6).

of the limit equations,* none of which satisfies the boundary conditions completely. The most attractive one is in general not the correct one.

The character of the possible limit flows is reminiscent of transsonic flow past a thin airfoil. As there the y-co-ordinate must be compressed, this time by a factor $\sqrt{\sigma H_0}$. Changes along the lines of force are thereby magnified and the new limit equations, which are fortunately still linear, take into account the inertia of the fluid. This leads to what may be called the outer expansions of the solution, one for y large and positive and the other for y large and negative. Their role is similar to, but somewhat simpler than, that of the Oseen expansion in slow viscous flow (Proudman & Pearson 1957) and, as there, they must be matched to an inner expansion (the Stokes expansion in the viscous case). There is only one such inner expansion, and its first two terms, which are solutions of the limit equations mentioned in the last paragraph, not only determine the correct limit flow at each point but also describe the way in which the pressure becomes infinite there.

It is easy to see how the compression factor $\sqrt{\sigma H_0}$ arises. For if changes in v take place over a distance of order R in the y-direction, then by continuity the disturbance velocity perpendicular to the field is of order $1/R$. This induces a ponderomotive force in the same direction of order $\sigma H_0^2/R$, which can only be balanced by pressure gradients and hence pressures of the same order. Thus pressure gradients along the lines of force of order $\sigma H_0^2/R^2$ occur and these can only be balanced by inertia forces, which are of order unity. Hence $R = \sqrt{\sigma H_0}$.

This also shows that the pressure and hence the forces on the cylinder are of order $\sqrt{\sigma H_0}$. The corresponding lift, drag, and moment are calculated for a flat plate at incidence and an elliptic cylinder.

The same analysis applies when the conductivity and magnetic field are both moderate, provided that then the free-stream velocity U_0 is small.† In this case our conclusions appear to be in contradiction to those of Stewartson (1956), who is forced to consider how the motion is set up from rest in attempting to determine the ultimate steady state. Although strictly speaking the two problems are not comparable—he considers a sphere of infinite conductivity while here a cylinder of zero conductivity‡ is taken—it would seem that his complete neglect of the quadratic inertia terms is not valid. Their rejection on the basis that the magnitude of the ponderomotive force is much larger is not justified for the motion along the magnetic field, since this force, however large, has no component in that direction. In fact the inertia forces play a critical role in the present treatment by dispersing the disturbance at large y-distances and thereby controlling the flow upstream and downstream of the cylinder.

We shall return to this point in a later paper, where the flow past a three-dimensional obstacle will be considered.

The author is indebted to S. Goldstein for valuable discussion during the preparation of this paper.

* This non-uniqueness was first recognized by Stewartson (1956).

† The compression factor $\sqrt{\sigma H_0}$ is replaced by $U_0^{-\frac{1}{2}}$. Note that now the dimensionless pressure is of orde1 $U_0^{-\frac{1}{2}}$ so that the actual forces on the cylinder are proportional to $U_0^{\frac{3}{2}}$.

‡ The present approximation does in fact also apply to a cylinder with a small conductivity.

2. The equations of motion

The fluid is assumed to be incompressible, inviscid and electrically conducting, so that its steady motion is governed by the equations

$$\left.\begin{aligned} \rho_0 \mathbf{v}.\operatorname{grad}\mathbf{v} &= -\operatorname{grad}p + \mu \operatorname{curl}\mathbf{H}\times\mathbf{H}, \\ \operatorname{curl}\mathbf{H} &= \sigma(\mathbf{E}+\mu\mathbf{v}\times\mathbf{H}), \end{aligned}\right\} \tag{1}$$

where $$\operatorname{div}\mathbf{v} = 0, \quad \operatorname{curl}\mathbf{E} = 0, \quad \operatorname{div}\mathbf{H} = 0. \tag{2}$$

Here $\mathbf{v}$ is the fluid velocity, ρ_0 the constant mass density, $\mathbf{E}$ and $\mathbf{H}$ the electric and magnetic fields, respectively, σ the electrical conductivity, and μ the permeability; the last two are taken to be constant throughout the fluid.

Take axes moving with the body, with Ox along the direction in which the fluid now moves at infinity, so that $\mathbf{v} = U_0\mathbf{i}$ there, and Oy along the undisturbed magnetic field $\mathbf{H}_0$.

Let a be a representative length in the body. Then, when $\mathbf{v}$, $\mathbf{r}$, and $\mathbf{H}$ are made dimensionless by referring them to U_0, a, and H_0, respectively, the equations (1) become

$$\mathbf{v}.\operatorname{grad}\mathbf{v} = -\operatorname{grad}p + \frac{1}{A^2}\operatorname{curl}\mathbf{H}\times\mathbf{H}, \tag{3a}$$

$$\operatorname{curl}\mathbf{H} = R_M(\mathbf{E}+\mathbf{v}\times\mathbf{H}), \tag{3b}$$

while equations (2) are unchanged. Here $A = U_0\sqrt{\rho_0}/\sqrt{\mu}H_0$ is the Alfvén number and $R_M = U_0 a\mu\sigma$ the magnetic Reynolds number; the electric field is now given by $(\mu U_0 H_0)\mathbf{E}$ and the pressure by $(\rho_0 U_0^2)p$. On substituting (3b) into (3a) we obtain the alternative form

$$\mathbf{v}.\operatorname{grad}\mathbf{v} = -\operatorname{grad}p + \frac{R_M}{A^2}(\mathbf{E}+\mathbf{v}\times\mathbf{H})\times\mathbf{H}. \tag{4}$$

In flows at low R_M, the disturbance of the applied magnetic field, due to the currents induced by the fluid motion, is seen to be determined by R_M [equation (3b)], while the influence of the field on the motion, through the force exerted by the field on these currents, is characterized by R_M/A^2 [see equation (4)]. This important parameter will be denoted by N:

$$N = \frac{R_M}{A^2} = \frac{a\mu^2 H_0^2\sigma}{\rho_0 U_0}. \tag{5}$$

When we say that the magnetic field is very strong we mean that N is large, even though R_M is small.

At large distances $\mathbf{E}$ tends to $-\mathbf{k}$. When the motion is plane, in the sense that $\mathbf{v}$ and $\mathbf{H}$ are independent of z and have no z-components, the second of equations (2) shows that $\mathbf{E}$ has this constant value throughout the flow.

3. Plane flow past a cylinder

We consider the flow past a cylinder whose generators are parallel to the z-axis, and proceed in a heuristic way, later checking the result obtained (§ 6).

As a first approximation for R_M small we may take curl $\mathbf{H} = 0$ [see (3b)]. Since the permeability of the cylinder is assumed to be the same as that of the fluid, this means that the magnetic field is undisturbed: $\mathbf{H} = \mathbf{j}$. When this and the result $\mathbf{E} = -\mathbf{k}$ are used in equation (4) it becomes

$$\left.\begin{aligned} \frac{\partial u}{\partial x}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x}-Nu, \\ \frac{\partial v}{\partial x}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y}, \end{aligned}\right\} \tag{6}$$

where u and v are the components of the disturbance velocity $\mathbf{v}-\mathbf{i}$.

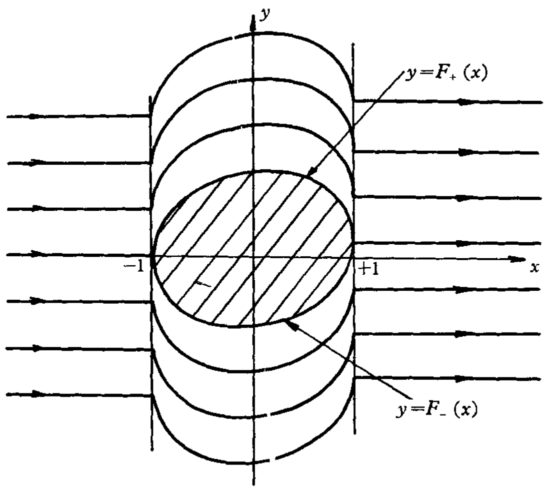

FIGURE 1. Plausible (but incorrect) solution of limit equations.

When N is large these equations reduce to

$$u = -\frac{\partial}{\partial x}\left(\frac{p}{N}\right), \quad \frac{\partial}{\partial y}\left(\frac{p}{N}\right) = 0, \tag{7}$$

if we allow for the possibility that p becomes large with N. Thus

$$p = Nf(x), \quad u = -f'(x),$$

and the equation of continuity gives

$$v = yf''(x)+g(x);$$

here f and g are arbitrary functions.

Although we shall not in fact be able to satisfy the boundary conditions completely, it is clear that $f''(x)$ must be set equal to zero. Then f' is a constant, which must be zero if u is to vanish at infinity; similarly, $f = 0$ since this is its value at infinity. There remains $g(x)$, which for $|x| \leqslant 1$* is determined by the boundary conditions at the cylinder $y = F_{\pm}(x)$ (see figure 1):

$$v = g(x) = \begin{cases} F'_{+}(x), & |x| \leqslant 1 \quad \text{and} \quad y \geqslant F_{+}(x), \\ F'_{-}(x), & |x| \leqslant 1 \quad \text{and} \quad y \leqslant F_{-}(x). \end{cases} \tag{8}$$

* The length a is henceforth taken to be one half the breadth of the cylinder in the x-direction.

This determination violates the conditions at infinity: v does not tend to zero as $y \to \pm\infty$. However, for $|x| > 1$, they can be satisfied by setting

$$v = g(x) = 0 \quad \text{for} \quad |x| > 1. \tag{9}$$

Then the violation is restricted to vanishingly small angles around the y-axis. Nevertheless, it will turn out that this is not in general the correct choice for $x < -1$.

This solution, which is apparently the only acceptable one, is illustrated in figure 1. The streamlines form two families of congruent curves. The flow is undisturbed until it comes abreast of the cylinder, where it is displaced to either side along the lines of force. Behind the cylinder it reunites into a uniform stream again. When, as in the figure, the cylinder is blunt at the front, infinite vertical velocities occur on the line $x = -1$. A similar remark applies to $x = 1$.

It would seem that this is the limiting form of the flow pattern in the vicinity of any fixed point as $N \to \infty$. However, when substituted into the neglected terms on the left-hand side of equations (6), it yields a perturbation in which, for $|x| < 1$, u and p are linear functions of y, and v is a quadratic one.

4. Compressing the y-co-ordinate

The main feature of the supposed limiting flow pattern is its frozen character in the vertical direction. Changes in this direction vanish in the limit at any fixed point. This suggests that, as this limit is being taken, the y-co-ordinate should be simultaneously compressed at a rate sufficient to retain significant changes. By reason of continuity, the disturbance velocity in the x-direction, which also vanishes in the limit, must be magnified at the same rate.

We therefore set

$$y = \sqrt{N}Y, \quad u = \frac{U}{\sqrt{N}}, \quad p = \sqrt{N}P \tag{10}$$

in equations (6), assume all the new variables and their derivatives are of order unity, and let N tend to infinity:

$$U = -\frac{\partial P}{\partial x}, \quad \frac{\partial v}{\partial x} = -\frac{\partial P}{\partial Y}. \tag{11}$$

Note that the inertia term $\partial v/\partial x$ has now survived [cf. (7)]. The choice (10) ensures that, in the new variables, the flow is the result of a balance between the inertia forces and the stress forces (both pressure and Maxwell).*

Eliminate U and P between equations (11) and the equation of continuity:

$$\frac{\partial U}{\partial x} + \frac{\partial v}{\partial Y} = 0. \tag{12}$$

Then

$$\frac{\partial^3 v}{\partial x^3} + \frac{\partial^2 v}{\partial Y^2} = 0. \tag{13}$$

* It is conceivable that the limit solution is the uniform flow and that this must be supplemented by a boundary layer at the cylinder. In the boundary layer the fluid would have to accelerate rapidly as it moved away from the front. However, the only force capable of such accelerations is the ponderomotive force and this would act in the wrong direction.

This equation will be solved separately above and below the x-axis, which is a characteristic. The two partial solutions must satisfy the boundary conditions

$$|x| \leqslant 1: \quad v = \begin{Bmatrix} F'_+(x) & \text{for} & Y \to +0, \\ F'_-(x) & \text{for} & Y \to -0, \end{Bmatrix} \tag{14}$$

see (8). For $|x| > 1$ they must lead to the same v and pressure P (and hence $\partial v/\partial Y$) on the x-axis.*

We may restrict our attention to the half plane $Y > 0$ (see figure 2), considering for the moment that v is prescribed everywhere on the x-axis. The problem is then quite similar to the initial-value problem of one-dimensional heat flow. Since the order of the derivatives in (13) is one higher, however, it would seem that the values of $\partial v/\partial Y$ also should be prescribed for $Y \to +0$. This is replaced by the requirement that v should vanish as $Y \to +\infty$, an automatic property in the heat case.†

The complete solution can be built up from the source solution, which is the x-derivative of the solution having the unit-step function for its boundary values. The latter corresponds to the error function of heat conduction and, like it, is a function of a single combination of x and Y.

5. The step-function solution

Consider the solution

$$v_0 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(\lambda Y - \lambda^{\frac{2}{3}} x)\frac{d\lambda}{\lambda} \quad (c > 0),$$

which can be obtained by use of the Laplace transform and is clearly a function of

$$\eta = \frac{x}{Y^{\frac{2}{3}}} \tag{15}$$

alone. The lines $\eta =$ const. are shown dashed in figure 2. In the first instance v_0 is defined for positive values of x only, but by deforming the path of integration in the complex λ-plane it can be continued analytically to all values of x (see Appendix).

For large values of $|\eta|$ it has the asymptotic expansions

$$v_0 \sim \frac{3}{2}\left[1 + \frac{1}{|\eta|^{\frac{3}{2}}}\sum_{n=0}^{\infty} \frac{1}{(2n+1)!\,\Gamma(-\frac{1}{2}-3n)}\frac{1}{|\eta|^{3n}}\right] \quad (\eta < 0), \tag{16a}$$

$$v_0 \sim \frac{9\exp[-4\eta^3/27]}{4(\pi\eta^3)}\sum_{n=0}^{\infty}\frac{A_n}{\eta^{3n}} \quad (\eta > 0), \tag{16b}$$

* See §6 for further clarification.

† Since the complete flow region is doubly connected the question of uniqueness of the solution arises, just as it does in the absence of a magnetic field [equations (6) with $N = 0$]. Although there is room to question uniqueness at this stage, where we are solving in the upper half plane assuming that v is prescribed on the x-axis, the real question comes later (end of next section), when we must determine v on the x-axis outside the body from the integral equation which results on joining this partial solution to a similar one in the lower half plane. However, it is known that this particular integral equation (Abel's) does have a unique solution.

where the coefficients A_n are determined by the recurrence relation

$$A_{n+1}+\tfrac{27}{4}(n+\tfrac{1}{2})\,A_n = \frac{\Gamma(\frac{3}{2})}{(2n+2)!\,\Gamma(-\frac{3}{2}-3n)}, \quad A_0 = 1. \tag{17}$$

Since v_0 tends to $\frac{3}{2}$ as $\eta \to -\infty$ and 0 as $\eta \to \infty$, it follows that

$$\mathscr{H}(\eta) = 1-\tfrac{2}{3}v_0 = 1-\frac{1}{3\pi i}\int_{c-i\infty}^{c+i\infty}\exp\,[\lambda Y-\lambda^{\frac{3}{2}}x]\,\frac{d\lambda}{\lambda}$$

is the required step-function solution.

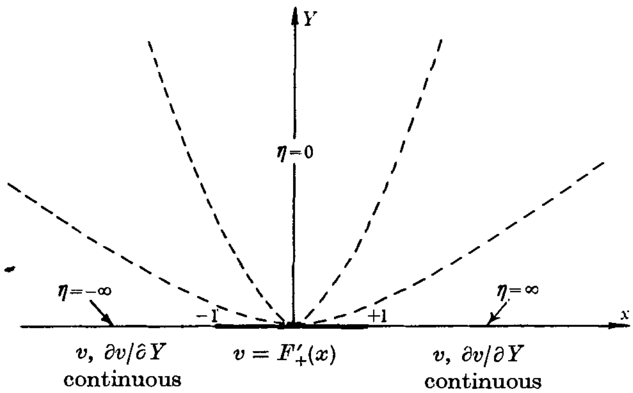

FIGURE 2. Boundary conditions on solution of equation (13) in upper half of x, Y-plane. On a dashed line the η of equation (15) has constant values.

A graph of $\mathscr{H}(\eta)$ is given in figure 3. For almost the whole range of η shown the series expansion

$$\mathscr{H}(\eta) = \tfrac{1}{3}+\tfrac{2}{3}\eta\sum_{n=0}^{\infty}\frac{(-1)^n}{(3n+1)!\,\Gamma(\frac{1}{3}-2n)}\eta^{3n}+\tfrac{2}{3}\eta^2\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(3n+2)!\,\Gamma(-\frac{1}{3}-2n)}\eta^{3n} \tag{18}$$

was found adequate for hand computations. The asymptotic expansions (16) confirmed values at the extremes. Note the gradual (algebraic) decay of $\mathscr{H}$ for negative η, determined by (16*a*), in contrast to the sharp (exponential) rise to its limiting value when η is positive, governed by (16*b*).*

The solution of (13) taking on the values $V(x)$ on the x-axis is therefore

$$v = \frac{1}{Y^{\frac{2}{3}}}\int_{-\infty}^{\infty} V(\xi)\,\mathscr{H}'[(x-\xi)/Y^{\frac{2}{3}}]\,d\xi, \tag{19}$$

since the x-derivative of $\mathscr{H}$ is clearly the source solution. The corresponding U and P now follow from (11) and (12). However, since we are mainly interested in the values of P as $Y \to +0$, which can be determined directly from the expansions (16), we shall not write the resulting formulas down. To satisfy (14) we set

$$V(x) = F'_+(x) \quad \text{for} \quad |x| \leqslant 1. \tag{20}$$

There remains the determination of V outside this range.

* For the case of heat conduction, the decay is exponential.

Consider the pressure P due to a unit source at the origin. Since

$$v = \frac{\partial \mathscr{H}}{\partial x} = -\frac{2}{3}\frac{\partial v_0}{\partial x},$$

we have

$$\frac{\partial^2 P}{\partial x^2} = \frac{\partial v}{\partial Y} = -\frac{2}{3}\frac{\partial^2 v_0}{\partial x \partial Y}.$$

Hence from the asymptotic expansions (16) we find that, as $Y \to +0$,

$$\frac{\partial^2 P}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{1}{2\sqrt{\pi}|x|^{\frac{3}{2}}}\right], \quad P = \frac{1}{\sqrt{\pi}|x|^{\frac{1}{2}}} \quad \text{for} \quad x < 0,$$

$$\frac{\partial^2 P}{\partial x^2} = 0, \quad P = 0 \quad \text{for} \quad x > 0,$$

since $P \to 0$ as $x \to \pm\infty$. This means that a source creates pressure upstream but not downstream of itself.

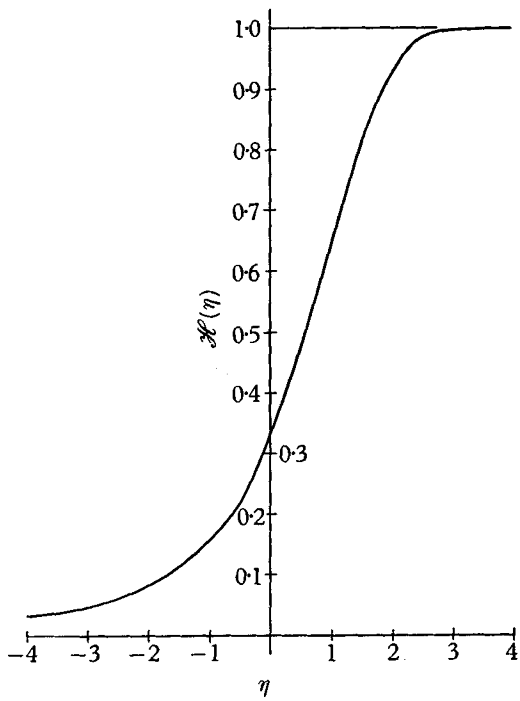

FIGURE 3. The step-function solution $\mathscr{H}(\eta)$ of equation (13), where $\eta = x/Y^{\frac{2}{3}}$.

It follows that

$$V(x) = 0 \quad \text{for} \quad x > 1 \tag{21}$$

in (19), and that correspondingly

$$P = \frac{1}{\sqrt{\pi}}\left[\int_x^{-1} \frac{V(\xi)}{(\xi - x)^{\frac{1}{2}}} d\xi + \int_{-1}^{1} \frac{F'_+(\xi)}{(\xi - x)^{\frac{1}{2}}} d\xi\right] \tag{22}$$

on the x-axis for $x < -1$. Similarly, for the solution in $Y < 0$ we have the values

$$P = -\frac{1}{\sqrt{\pi}}\left[\int_x^{-1} \frac{V(\xi)}{(\xi - x)^{\frac{1}{2}}} d\xi + \int_{-1}^{1} \frac{F'_-(\xi)}{(\xi - x)^{\frac{1}{2}}} d\xi\right], \tag{23}$$

where the same function V occurs by reason of the continuity of v across the x-axis.

Now equate the right-hand sides of (22) and (23). The result is an Abel integral equation for V, whose unique solution* is

$$V(x) = \frac{1}{2\pi}\frac{d}{dx}\int_x^{-1}\frac{d\eta}{(\eta-x)^{\frac{1}{2}}}\int_{-1}^{1}[F'_+(\xi)+F'_-(\xi)]\frac{d\xi}{(\xi-\eta)^{\frac{1}{2}}}$$

$$= -\frac{1}{2\pi}\frac{1}{\sqrt{(-1-x)}}\int_{-1}^{1}[F'_+(\xi)+F'_-(\xi)]\frac{(1+\xi)^{\frac{1}{2}}}{\xi-x}\,d\xi \quad \text{for} \quad x < -1. \quad (24)$$

6. Discussion of solution: the force on the cylinder

We must now indicate in a more precise way the character of the solution which has been obtained. What is in fact involved in a matching procedure similar to those in boundary-layer theory [see Goldstein (1960, p. 131 *et seq.*)] and slow viscous motion [see Proudman & Pearson (1957)], though of a simpler kind.

The variables u, v, p as functions of x, y will be called the inner solution, while U, v, P as functions of x, Y, for $Y > 0$ and $Y < 0$ separately, will be referred to as the outer ($\pm$) solutions. Correspondingly, we shall speak of the inner layer and outer regions.

In §3 we showed, by plausible arguments, that the first approximation to the inner solution for large values of N was

$$u = 0, \quad v = \begin{cases} F'_\pm(x) & \text{for} \quad |x| \leqslant 1, \\ g(x) & \text{for} \quad |x| > 1, \end{cases} \quad p = O(N), \quad (25)$$

correct to $O(1)$, $O(1)$, and $O(N)$ respectively, where now we shall not take $g = 0$. In §4 we assumed that the outer solutions would have to satisfy these conditions (translated into U, v, P-variables) as $Y \to \pm 0$. For $|x| \leqslant 1$ this prescribes definite (but in general different) values for v as $Y \to \pm 0$. For $|x| \geqslant 1$ it merely requires the values to be the same, $g(x)$. However, since in the outer regions a velocity v induces a pressure P of the same order, a pressure p of order $\sqrt{N}$ must be expected in the inner layer, and such a pressure is transmitted unaltered across the layer. This leads then to the additional requirement that the P of the outer solutions should tend to the same values as $Y \to \pm 0$, for $|x| > 1$.†

Thus the disturbance created by the cylinder transmits itself through the inner layer along the magnetic lines of force into the outer regions, where it disperses and thereby influences conditions upstream (but not downstream) in the inner layer.

The first approximation to the outer (+) solution was determined in the last section. The velocity v is given by (19) where V has the values (20), (21) and (24). Similar formulas hold for the outer (−) solution. This determines not only g,

$$g(x) = V(x) \quad (|x| > 1),$$

but also most of the next approximation in the inner solution. For, in order to provide appropriate $\sqrt{N}$-terms, the latter must be a solution of (7), rather than

* See second footnote, p. 146.

† There is a similar requirement on U but this is satisfied automatically [see (11)].

the result of perturbing (25) by means of (6); consequently it may be written in the form

$$u = -\frac{1}{\sqrt{N}}P_0'(x), \quad v = V(x)+\frac{1}{\sqrt{N}}\begin{cases}[y-\tilde{F}_\pm(x)]P_0''(x) & \text{for} \quad |x| \leqslant 1,\\ [y-h(x)]P_0''(x) & \text{for} \quad |x| > 1,\end{cases}$$
$$p = \sqrt{N}P_0(x). \tag{26}$$

The choice
$$\tilde{F}_\pm(x) = F_\pm(x)+\frac{P_0'(x)}{P_0''(x)}F_\pm'(x)$$

ensures that the boundary condition at the cylinder is satisfied to order $1/\sqrt{N}$. Also, setting $P_0(x)$ equal to either of the expressions (22) and (23) for $x < -1$, or the equivalent expression
$$\frac{1}{2\sqrt{\pi}}\int_{-1}^{1}\frac{F_+'(\xi)-F_-'(\xi)}{(\xi-x)^{\frac{1}{2}}}d\xi; \tag{27}$$

zero for $x > 1$; and
$$\pm\frac{1}{\sqrt{\pi}}\int_x^1\frac{F_\pm'(\xi)}{(\xi-x)^{\frac{1}{2}}}d\xi \quad \text{for} \quad |x| < 1; \tag{28}$$

makes the pressure p join smoothly with that in the outer regions.

As was the case for $g(x)$ in the approximation (25), $h(x)$ is left undetermined. It has to be found from the common limit for v in the next approximation to the outer solutions as $Y \to \pm 0$ with $|x| > 1$. Note that the y-terms in (26) already appear in the expansions (small Y) of the previous approximation to the outer solutions, and that any part $k(x)$ of the functions $\tilde{F}_\pm$ and h can be absorbed by them if we now set
$$Y = [y-k(x)]/\sqrt{N} \tag{10'}$$

[cf. (10)]. This appears to be of some importance since otherwise, for example, unmanageably singular data.

$$v = F_\pm'(x)-\frac{1}{\sqrt{N}}\tilde{F}_\pm(x)P_0''(x) \quad \text{as} \quad Y \to \pm 0 \quad \text{with} \quad |x| \leqslant 1$$

occur for this next approximation (see next section). The pressure P of order $1/\sqrt{N}$ which is introduced must once again be continuous across $Y = 0$, as is easily checked by determining the form of the term of order unity in p for the inner solution [from (6)].

The next approximation to the outer solutions involves a perturbation of the previous approximation, and cannot be expressed in simple form. In any event, it is not needed in order to obtain the pressure p, and hence the forces on the cylinder, to order $\sqrt{N}$ [see (26)]. We find from (28) the following lift, drag, and moment.

$$\left.\begin{aligned}
\frac{L}{\rho_0 U_0^2 a} &= -\sqrt{\frac{N}{\pi}}\int_{-1}^{1}dx\int_x^1\frac{F_+'(\xi)+F_-'(\xi)}{(\xi-x)^{\frac{1}{2}}}d\xi\\
&= -2\sqrt{\frac{N}{\pi}}\int_{-1}^{1}(\xi+1)^{\frac{1}{2}}[F_+'(\xi)+F_-'(\xi)]d\xi,\\
\frac{D}{\rho_0 U_0^2 a} &= \sqrt{\frac{N}{\pi}}\int_{-1}^{1}dx\int_x^1\frac{F_+'(x)F_+'(\xi)+F_-'(x)F_-'(\xi)}{(\xi-x)^{\frac{1}{2}}}d\xi,\\
\frac{M}{\rho_0 U_0^2 a^2} &= -\sqrt{\frac{N}{\pi}}\int_{-1}^{1}dx\int_x^1\{[F_+(x)F_+'(x)+x]F_+'(\xi)\\
&\qquad +[F_-(x)F_-'(x)+x]F_-'(\xi)\}\frac{d\xi}{(\xi-x)^{\frac{1}{2}}}.
\end{aligned}\right\} \tag{29}$$

7. Examples: flow past a flat plate and elliptic cylinder

Some general conclusions can be drawn from the formulas. From (24) we find

$$\int_{-\infty}^{-1} V(x)\,dx = -\frac{1}{2}\int_{-1}^{1} [F'_+(\xi)+F'_-(\xi)]\,d\xi = d,$$

where $$d = F_+(-1)-F_+(1) = F_-(-1)-F_-(1)$$

is the downward displacement of the rear point of the profile with respect to its front point. Since the dividing streamline is straight behind the profile, this means that the total displacement of this streamline is zero in the first approximation (see figure 4).

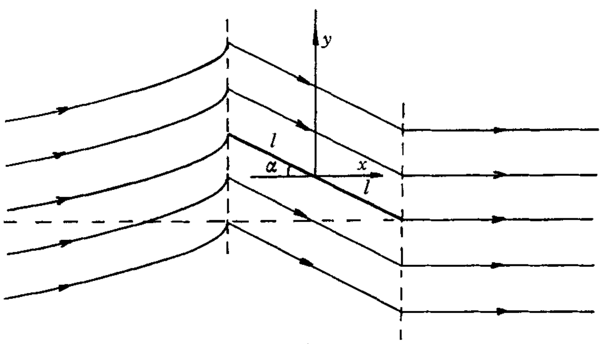

FIGURE 4. Limiting form of flow past a flat plate at incidence.

The pressure upstream of the profile depends only on the values of $F_+(x)-F_-(x)$, i.e. on the thickness in the y-direction but not on the position of the mean surface $y = \frac{1}{2}[F_+(x)+F_-(x)]$ [see (27)]. On the other hand, the lift depends on the mean surface but not on the thickness.

(*a*) *Flat plate.* Let the breadth be $2l$ and the angle of attack α (figure 4). Then $F_\pm = -x\tan\alpha$, $a = l\cos\alpha$ and

$$V = \frac{2\tan\alpha}{\pi}\left[\sqrt{\left(\frac{2}{-1-x}\right)} - \tan^{-1}\sqrt{\left(\frac{2}{-1-x}\right)}\right] \quad \text{for} \quad x < -1,$$

$$P_0 = \mp\frac{2\tan\alpha}{\pi}\sqrt{(1-x)} \quad \text{for} \quad |x| < 1.$$

Elsewhere P_0 is zero, in accordance with the remarks just made. The forces on the plate are

$$L = \frac{16}{3}\sqrt{\left(\frac{2N}{\pi}\right)}\rho_0 U_0^2 l\sin\alpha = 4{\cdot}255\sqrt{N}\rho_0 U_0^2 l\sin\alpha,$$

$$D = \frac{16}{3}\sqrt{\left(\frac{2N}{\pi}\right)}\rho_0 U_0^2 l\sin\alpha\tan\alpha = 4{\cdot}255\sqrt{N}\rho_0 U_0^2 l\sin\alpha\tan\alpha,$$

with a resultant perpendicular to the plate; while the moment is

$$M = -\frac{16}{15}\sqrt{\left(\frac{2N}{\pi}\right)}\rho_0 U_0^2 l\tan\alpha = -0{\cdot}851\sqrt{N}\rho_0 U_0^2 l\tan\alpha,$$

so that the resultant acts at a point $l/5$ forward of the centre.

(*b*) *Elliptic cylinder.* Let the major axis be horizontal and of length $2a$. Then $F_{\pm} = \pm (b/a)\sqrt{(1-x^2)}$, where $2b$ is the length of the minor axis. From the symmetry,
$$V = 0 \quad \text{for} \quad x < -1, \quad \text{and} \quad L = M = 0.$$
We find
$$P_0 = \begin{cases} \dfrac{2}{\sqrt{\pi}}\dfrac{b}{a}\sqrt{(1-x)}\left[\dfrac{-x}{1-x}K\left(\sqrt{\dfrac{2}{1-x}}\right) - E\left(\sqrt{\dfrac{2}{1-x}}\right)\right] & \text{for} \quad x < -1, \\ \sqrt{\dfrac{2}{\pi}}\dfrac{b}{a}\left[K\left(\sqrt{\dfrac{1-x}{2}}\right) - 2E\left(\sqrt{\dfrac{1-x}{2}}\right)\right] & \text{for} \quad |x| < 1, \end{cases}$$
where K and E are complete elliptic integrals of the first and second kind, respectively. Finally, the drag is
$$D = \frac{1}{\sqrt{(18\pi)}} B^2(\tfrac{1}{4}, \tfrac{1}{2})\frac{b^2}{a^2}\sqrt{N\rho_0}\,U_0^{\frac{3}{2}}a = 3{\cdot}656\frac{b^2}{a^2}\sqrt{N\rho_0}\,U_0^{\frac{3}{2}}a,$$
where B is the beta function.

A circular cylinder of diameter $2a$ experiences a drag equal to that of a flat plate of breadth $2a$ inclined at an angle α of about 49°.

Finally, we note that the function $\bar{F}_{\pm}(x)\,P_0''(x)$ which occurs in equation (26) is infinite of order $\frac{3}{2}$ as $x \to 1$ in (*a*) and as $x \to -1$ in (*b*). Both these singularities can be avoided in the boundary data for the second approximation to the outer solutions by suitable non-singular choices of $k(x)$ in (10′). Whether this leads to an acceptable approximation remains an open question. In any event the case of the elliptic cylinder, which is bluff both forward and aft, appears to be no more singular than that of the flat plate.

These successively worse singularities on $x = \pm 1$ seem to form an integral part of the expansion of the inner solution, since they arise from matching with expansions of the outer solutions, which in turn derive from essentially asymptotic representations of the type (16).

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Appendix

Here we derive the properties of the function
$$v_0 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp\left[\lambda Y - \lambda^{\frac{3}{2}}x\right]\frac{d\lambda}{\lambda} \quad (c > 0) \tag{30}$$
quoted in the text (see figure 5).

For $x > 0$, the integral is convergent and the path of integration may be deformed into $ABCD$ starting at $-\infty$, encircling the origin in the anti-clockwise direction, and returning to $-\infty$. Thus, for $0 \leqslant \arg\lambda < 3\pi/4$, $|\exp(-\lambda^{\frac{3}{2}}x)|$ is less than unity, while, for $3\pi/4 \leqslant \arg\lambda \leqslant \pi$, $\exp\lambda Y$ is dominantly small in the integrand, i.e. $|\exp(\lambda Y - \lambda^{\frac{3}{2}}x)| < \exp(-m|\lambda|)$ for any positive $m < Y/\sqrt{2}$. A slight modification of the usual argument (Carslaw & Jaeger 1948, p. 76) now shows that the integral taken over the infinite circular arc Q_1 is zero. Similar remarks apply to Q_2. Hence
$$v_0 = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} \exp\left[\lambda Y - \lambda^{\frac{3}{2}}x\right]\frac{d\lambda}{\lambda}, \tag{31}$$
the integral now converging for all values of x.

If $\exp(-\lambda^{\frac{2}{3}}x)$ is expanded and term-by-term integration is made [a procedure which is valid for (31) but not (30)], we obtain*

$$
\begin{aligned}
v_0 &= \frac{1}{2\pi i}\int_{-\infty}^{(0+)} \frac{e^{\lambda Y}}{\lambda}\left[1 - x\lambda^{\frac{2}{3}} + \frac{x^2\lambda^{\frac{4}{3}}}{2!} - \frac{x^3\lambda^2}{3!} + \dots\right] d\lambda \\
&= 1 - \frac{x}{\Gamma(\frac{1}{3})\, Y^{\frac{2}{3}}} + \frac{x^2}{2!\Gamma(-\frac{1}{3})\, Y^{\frac{4}{3}}} + \dots \\
&= 1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)!\,\Gamma(\frac{1}{3}-2n)}\left(\frac{x}{Y^{\frac{2}{3}}}\right)^{3n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+2)!\,\Gamma(-\frac{1}{3}-2n)}\left(\frac{x}{Y^{\frac{2}{3}}}\right)^{3n+2},
\end{aligned}
$$

from which follows (18) [see (15)]. These series converge (exponentially) for all values of $\eta = x/Y^{\frac{2}{3}}$.

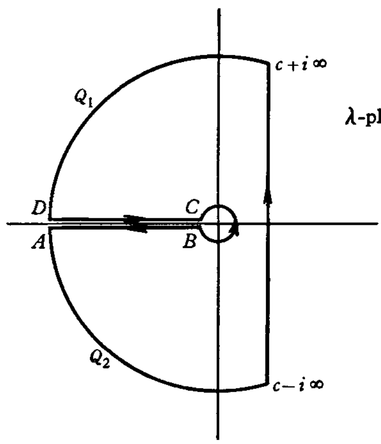

FIGURE 5. Paths of integration in λ-plane for (30) and (31).

On the other hand, when x is restricted to negative values we may expand $\exp \lambda Y$ and integrate term by term, thus obtaining the asymptotic representation for η large and negative†

$$
\begin{aligned}
v_0 &= \frac{1}{2\pi i}\int_{-\infty}^{(0+)} \frac{e^{-\lambda^{\frac{2}{3}}x}}{\lambda}\left[1 + Y\lambda + \frac{Y^2\lambda^2}{2!} + \frac{Y^3\lambda^3}{3!} + \dots\right] d\lambda \\
&= \frac{3}{2}\left[1 + \frac{Y}{\Gamma(-\frac{1}{2})\,(-x)^{\frac{3}{2}}} + \frac{Y^3}{3!\Gamma(-\frac{7}{2})\,(-x)^{\frac{9}{2}}} + \dots\right] \\
&= \frac{3}{2}\left[1 + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!\,\Gamma(-\frac{1}{2}-3n)}\left(\frac{Y^{\frac{2}{3}}}{-x}\right)^{3n+\frac{3}{2}}\right].
\end{aligned}
$$

Here we have used the transformation $\kappa = \lambda^{\frac{2}{3}}$ and have noticed that the resulting path of integration in the κ-plane can be deformed into one similar to that in the λ-plane. For $m = 0, 1, 2, \dots$

$$
\int_{-\infty}^{(0+)} e^{-\lambda^{\frac{2}{3}}x}\,\lambda^m\, d\lambda = \frac{3}{2}\int_{-\infty}^{(0+)} e^{\kappa(-x)}\,\kappa^{\frac{1}{2}(3m+1)}\, d\kappa = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ \dfrac{3}{2}\dfrac{2\pi i}{\Gamma(-\frac{1}{2}-\frac{3}{2}m)\,(-x)^{3(m+1)/2}} & \text{if } m \text{ is even.} \end{cases}
$$

Thus we obtain (16a).

* See, for example, Jeffreys (1927, p. 23 *et seq.*).

† In the heat-conduction case, where $\lambda^{\frac{2}{3}}$ is replaced by $\lambda^{\frac{1}{2}}$, this does not work: the function is exponentially small.

To obtain the asymptotic representation of v_0 when x is positive and η is large we set $\lambda = x^3z^3/Y^3$ in (31). With $k = x^3/Y^2$ ($= \eta^3$),

$$v_0 = \frac{3}{2\pi i}\int_{(1-\sqrt{3}i)\infty}^{(1+\sqrt{3}i)\infty} \exp\left[k(z^3-z^2)\right]\frac{dz}{z}, \tag{32}$$

where the path of integration passes to the right of the origin (see figure 6). The integral is now in standard form for applying the method of steepest descent (Copson 1946; Erdélyi 1956).

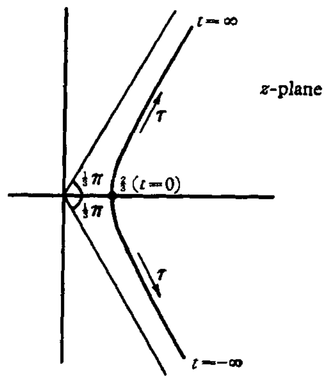

FIGURE 6. The steepest descent paths, leading from the col $z = \frac{2}{3}$, used in finding the asymptotic expansion of (32) for large k.

There are two cols, $z = 0$ and $z = \frac{2}{3}$. The first is not acceptable; through the second passes the steepest descent curve

$$3x^2 - y^2 - 2x = 0.$$

Since the right branch of this hyperbola tends to infinity in the directions $\arg z = \pm\frac{1}{3}\pi$, it may be used as the path of integration in (32). Let

$$z^3 - z^2 = -\tfrac{4}{27} - t^2,$$

so that on the hyperbola t is real and increases monotonically from $-\infty$ through 0 (at $z = \frac{2}{3}$) to $+\infty$. Then we must determine $\log z$ as a function of t [since $dz/z = d(\log z)$].

Let

$$\log z = \log\tfrac{2}{3} + W;$$

then

$$\frac{2}{3\sqrt{3}}\,W\left(\frac{e^W-1}{W}\right)(2e^W+1)^{\frac{1}{2}} = it,$$

where $(2e^W+1)^{\frac{1}{2}}$ denotes the branch which reduces to $\sqrt{3}$ when $W = 0$ (i.e. $t = 0$). By Lagrange's formula for the reversion of series (Copson 1935) the solution is

$$W = \sum_{n=1}^{\infty} B_n(it)^n,$$

where $$nB_n = \text{residue of}\left[\frac{2}{3\sqrt{3}}W\left(\frac{e^W-1}{W}\right)(2e^W+1)^{\frac{1}{2}}\right]^{-n} \quad \text{at} \quad W=0$$
$$= \text{residue of}\,(1+w)^{-1}\left[\frac{2}{3\sqrt{3}}w\,(2w+3)^{\frac{1}{2}}\right]^{-n} \quad \text{at} \quad w=0^*$$
$$= \text{sum of the first } n \text{ coefficients in } (-1)^{n-1}\left(\tfrac{3}{2}\right)^n\left(1-\tfrac{2}{3}w\right)^{-\frac{1}{2}n}$$
$$= (-1)^{n-1}\left(\tfrac{3}{2}\right)^n\left[3^{\frac{1}{2}n}-\left(\tfrac{2}{3}\right)^n\frac{\Gamma(\frac{3}{2}n)}{\Gamma(\frac{1}{2}n)\,\Gamma(n+1)}F\left(\tfrac{3}{2}n,1,n+1;\tfrac{2}{3}\right)\right].$$

The integral (32) may now be written in terms of t. Then if we set $t=\sqrt{\tau}$ for $t>0$ and $t=-\sqrt{\tau}$ for $t<0$ it becomes

$$v_0 = \frac{3}{4\pi i}\exp\left[-\frac{4k}{27}\right]\int_0^\infty e^{-k\tau}\left[\sum_{n=1}^\infty\{i^n-(-i)^n\}\,nB_n\tau^{\frac{1}{2}(n-1)}\right]\frac{d\tau}{\sqrt{\tau}}$$
$$= \frac{3}{2\pi}\exp\left[-\frac{4k}{27}\right]\sum_{m=0}^\infty(-1)^m(2m+1)\,\Gamma(m+\tfrac{1}{2})\,B_{2m+1}k^{-(m+\frac{1}{2})}$$
$$= \frac{9\exp\left[-\frac{4}{27}k\right]}{4\sqrt{(\pi k)}}\sum_{m=0}^\infty A_m k^{-m},$$

where

$$A_m = \frac{4}{3\sqrt{\pi}}(-1)^m\,\Gamma(m+\tfrac{3}{2})\,B_{2m+1}$$
$$= \frac{2}{3\sqrt{\pi}}(-1)^m\,\Gamma(m+\tfrac{1}{2})\left[\left(\tfrac{27}{4}\right)^{m+\frac{1}{2}}-\frac{\Gamma(3m+\frac{3}{2})}{\Gamma(m+\frac{1}{2})\,\Gamma(2m+2)}F\left(3m+\tfrac{3}{2},1,2m+2;\tfrac{2}{3}\right)\right].$$

Since $k=\eta^3$ this is (16*b*). The recurrence relation (17) easily follows, and can be checked by direct substitution of (16*b*) into the differential equation (13).

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* Setting $W=\log(1+w)$ in the contour-integral representation of the residue.